Notes on Scattering and Mobility in 1D, 2D, and 3D

These notes on carrier scattering in 1D, 2D, and 3D were written in the fall of 2005 by Drs. Dmitri Nikonov and George Bourianoff of Intel Corporation and by Dr. Sayed Hasan, formerly a Ph.D. student at Purdue University and now also at Intel Corporation. Lecture 26 in ECE -656, fall 2009 at Purdue University is based on these notes. The course lectures are available online at:

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Mobility in nanowires: Step by step

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Abstract

Derivation of the phonon-limited mobility is reviewed for electrons in bulk (3D) or quantum confined (2D and 1D) semiconductor structures. Analytical estimates are made that show the mobility in quantum confined structures is, in general, lower or no higher than in non-confined ones.

Motivation

This note deals with the comparison of electron mobility in traditional CMOS transistors and in nanowire transistors. It is commonly assumed that mobility is increased in nanowire transistors due to a decreased density of states for electrons available for scattering. However, a recent simulation result of Intel TCAD team [1] contradicts this expectation.

This note is intended to provide a pedagogical review of the mobility calculation. Where possible we make simplifying approximations in order to uncover the physical reasons for change of mobility in quantum confined structures. This discussion leaves out the effects of scattering from impurities of imperfections in the crystal or interface as well as electron-electron scattering. Quantum confined structures may improved these contributions to mobility.

Quantum mechanics of electron-phonon interaction

At room temperature the mobility of charges in semiconductor devices is primarily limited by phonon scattering. In the following, we review the description of the interaction of free carriers in a semiconductor with phonons in a form that is valid for cases of unconfined electrons (3D, bulk), or confined in one dimension (2D, quantum well), or in two dimensions (1D, nanowire). In this note, D = number of unconfined dimensions, L = number of confined dimensions.

We consider for simplicity only electrons in the conduction band (rather than holes in the valence band). As it is the case in silicon, electrons can exist in a number of valleys v_v (= minima of energy) of the conduction band. In silicon, the six degenerate valleys with lowest energy are dominant. For simplicity, we assume the conduction band parabolic

and isotropic with mass m in the vicinity of the valleys. It was found [1] that non-isotropic bands result in somewhat lower mobility for confined structures. We disregard the effect of the confining structure on the spectrum of phonons (i.e., still consider them uniform in 3D). It was found [2] that the use of bulk phonons also slightly decreases that value of mobility.
We designate the coordinates in the unconfined dimensions as r and in confined dimensions as z, so that the three-dimensional vector is $R=(r,z)$. The states of electrons are labeled by their momentum in the unconfined dimensions and by the subband index in the confined dimensions $K=(k,n)$. The states of the phonons are labeled by their momentum with components in the confined and unconfined dimensions $Q=(q,\tilde{q})$. The general form of the electron-phonon interaction Hamiltonian is

$$H=\sum_{Q} M(Q)(c_Q+c_{-Q}^{+})\exp(iQR), \tag{1}$$

where c_Q, c_Q^{+} are the annihilation and creation operators for a phonon in mode Q, and $M(Q)$ is the matrix element of interaction.

Intravalley scattering is mediated by acoustic phonons and the intervalley scattering by both acoustic and optical phonons [3]. The matrix element for the longitudinal acoustic phonons is [4]

$$M(Q)_{ac}=\Xi Q\sqrt{\frac{\hbar}{2L^3\rho\omega(Q)}}, \tag{2}$$

where Ξ is the deformation potential (=12eV for Si), L is the nominal size of the semiconductor in the unconfined dimensions, ρ is the density of the material, in the following we assume the frequency of acoustic phonons proportional to momentum $\omega=u_l Q$ with the constant of longitudinal speed of sound.

For intervalley transitions (optical phonons) the matrix element is [4]

$$M(Q)_{iv}=D_{iv}\sqrt{\frac{\hbar}{2L^3\rho\omega_r}}, \tag{3}$$

where D_{iv} is the optical coupling constant ($D_{iv}\sim 10^{11}$eV/m, see [4] for details), ω_r is the frequency of optical phonons (often set constant, =36meV in g-processes and 54meV in f-processes for Si). Most of the further derivation is common to all types of phonon-assisted processes.

We will calculate the rate of scattering using the "Fermi Golden Rule" in the form of [4]. The probability per unit time to scatter from state K to state K' is

$$P(K,c,K',c')=\frac{2\pi}{\hbar}\left|\langle K'c'|H|Kc\rangle\right|^2\delta(E(K',c')-E(K,c)), \tag{4}$$

where c, c' stand for the state of the crystal with the number of phonons in each momentum state before and after scattering, and E is the energy of a state.

The envelope wavefunctions (which are orthogonal and normalized) for the states of electron are products of the plane wave and the confined wavefunction

$$\langle r, z | k, n \rangle = \frac{1}{L^{D/2}} \exp(ikr) u_n(z). \tag{5}$$

The summation over the states of electrons is defined as

$$\sum_K = \sum_n \int \frac{L^D d^D k}{(2\pi)^D}. \tag{6}$$

One also needs to multiply by the number of equivalent valleys ν_r. The summation over the coordinates (implied in the bracket expression of the transition element in (4)) is defined as

$$\sum_R = \int \frac{d^D r}{L^D} \int dz. \tag{7}$$

Thus, the transition amplitude

$$\langle K'c' | H | Kc \rangle = L^D \sum_{Q,R} \langle k', n' | r, z \rangle \langle r, z | k, n \rangle \exp(iqr + i\tilde{q}z) M(Q) \langle c' | c_Q + c_{-Q}^+ | c \rangle. \tag{8}$$

The matrix elements of the creation and annihilation operators are given by the number of phonons N_Q in the mode [3],

$$\langle c' | c_Q | c \rangle = \sqrt{N_Q}, \qquad \langle c' | c_{-Q}^+ | c \rangle = \sqrt{N_Q + 1}, \tag{9}$$

correspond to the scattering processes with absorption and emission of a photon, respectively. The coordinate integration results in the expression for momentum conservation, $k' = k \pm q$, in unconfined dimensions. Here and later in this note, the upper sign in the expression will correspond to absorption and the lower to emission of a phonon.

$$\langle K'c' | H | Kc \rangle = \sum_Q \delta_{k', k \pm q} F_{n'n}(\tilde{q}) M(Q) \sqrt{N_Q + \frac{1}{2} \mp \frac{1}{2}}. \tag{10}$$

In Eq. (1), the form-factor of confinement is

$$F_{n'n}(\tilde{q}) = \int u_{n'}^*(z) \exp(i\tilde{q}z) u_n(z) dz. \tag{11}$$

The corresponding conservation of energy is expressed in the delta function in (4) such that the transition rate becomes (with implied momentum conservation)

$$\boxed{P^{a,e}(K,K') = \frac{2\pi}{\hbar}\sum_{\tilde{q}}\left|F_{n'n}(\tilde{q})\right|^2\left|M(Q)\right|^2\left(N_Q+\frac{1}{2}\mp\frac{1}{2}\right)\delta(E(k',n')-E(k,n)\mp\hbar\omega(Q))}. \quad (12)$$

Density of states with quantum confinement

In order to calculate the density of states over energy, one has to transform the expression for the number of states in *D* unconfined dimensions in one subband with the account of two possible spin states,

$$N_{st} = 2\int\frac{L^D d^D k}{(2\pi)^D} = L^D\int_0^\infty \mathcal{D}^{(D)}(E)dE. \quad (13)$$

In Eq. (13) we for the first time explicitly use the parabolic approximation to energy in the bands

$$E = \frac{\hbar^2 k^2}{2m}. \quad (14)$$

Next, after simple algebra, one derives the expressions for the density of states for various dimensions of the electron states

$$\mathcal{D}^{(3)}(E) = \frac{\sqrt{2m^3E}}{\pi^2\hbar^3} = \frac{mk}{\pi^2\hbar^2}. \quad (15)$$

$$\mathcal{D}^{(2)}(E) = \frac{m}{\pi\hbar^2}. \quad (16)$$

$$\mathcal{D}^{(1)}(E) = \frac{\sqrt{2m}}{\pi\hbar\sqrt{E}} = \frac{2m}{\pi\hbar^2 k}. \quad (17)$$

Scattering rates

To obtain useful expressions for the scattering rate, we make more simplifications. The form factor has the strongest dependence on the transverse momentum. Therefore, the sum over the phonon momentum results in the overlap factor

$$\sum_{\tilde{q}}\left|F_{n'n}(\tilde{q})\right|^2 = \int\frac{L^L d^L\tilde{q}}{(2\pi)^L}\left|F_{n'n}(\tilde{q})\right|^2 = L^L\int\left|u_{n'}^*(z)\right|^2\left|u_n(z)\right|^2 dz \equiv L^L\mathcal{F}_{n'n}. \quad (18)$$

Next, for acoustic phonons at high temperature, we approximate their number of phonons per mode as

$$N(Q)+\frac{1}{2} \approx \frac{k_B T}{\hbar\omega(Q)}. \quad (19)$$

We also neglect the energy of a phonon compared to the energy of the electron (quasi-elastic approximation). Therefore the total emission rate (with both emission and absorption) for acoustic phonons is

$$P_{ac}^{a+e}(k,n,k',n') = \frac{2\pi}{\hbar}\frac{\mathcal{F}_{n'n}}{L^D}\frac{k_B T \Xi^2}{\rho u_l^2}\delta(E(k',n') - E(k,n)) . \tag{20}$$

For intervalley transitions, we set the energy and the number of phonons per mode constant for all transitions. The separate rates for absorption or emission are

$$P_{iv}^{a,e}(k,n,k',n') = \frac{2\pi}{\hbar}\frac{\mathcal{F}_{n'n}}{L^D}\frac{D_{iv}^2\hbar\left(N_r + 1/2 \mp 1/2\right)}{2\rho\omega_r}\delta(E(k',n') - E(k,n) \mp \hbar\omega_r) . \tag{21}$$

The acoustic and intervalley phonon transition rates have very similar expressions; both are isotropic relative to final states under the above approximations. They both have the form

$$P(k,n,k',n') = U\frac{\mathcal{F}_{n'n}}{L^D}2\delta(E(k',n') - E(k,n) \mp \hbar\omega_r) . \tag{22}$$

The prefactors for intravalley transitions are

$$U_{ac} = \frac{\pi}{\hbar}\frac{k_B T \Xi^2}{\rho u_l^2} \sim 3\times 10^{-34}\,\mathrm{J}\bullet\mathrm{m}^3/\mathrm{s} . \tag{23}$$

and for intervalley transitions

$$U_{iv}^{a,e} = \frac{\pi}{\hbar}\frac{D_{iv}^2\hbar\left(N_r + 1/2 \mp 1/2\right)}{2\rho\omega_r} \sim 1.2\times 10^{-33}\,\mathrm{J}\bullet\mathrm{m}^3/\mathrm{s} . \tag{24}$$

For silicon, the intervalley scattering processes turns out to be more dominant due to the factor of number of valleys. By separating the prefactors, we can focus on the effects of low-dimensional density of states on mobility.

Finally, we can calculate the total scattering rate – summed over the final momenta of electrons. The summation is performed very easily if we replace it by integration over energy with the factor of density of states under the integral. The density of states contains the factor of 2 due to summation over the spin of electrons, therefore we need to insert an extra factor of ½ .

$$P(k,n,n') = \sum_{k'} P(k,n,k',n') = \int dE'\mathcal{D}^{(D)}(E')\frac{1}{2}P(k,n,k',n') . \tag{25}$$

The delta-function containing the energies before and after the collision brings out the electron density of states per unit energy $\mathcal{D}^{(D)}(E')$ in the final state. This density of states is calculated in a single valley in a specific subband.

$$P(k,n,n') = \mathcal{D}^{(D)}(E')\mathcal{F}_{n'n}U\,. \tag{26}$$

To calculate conductivity we will need not the collision rate, but the collision momentum relaxation rate, containing the scalar product of momentum vectors,

$$\tilde{P}(k,n,n') = \sum_{k'} \frac{(k-k')\bullet k}{k^2} P(k,n,k',n')\,. \tag{27}$$

In this equation all the momenta have an origin at the bottom of the corresponding valley and therefore correspond to velocities. One then should not forget to include constant momenta change in intervalley transitions. The meaning of the momentum relaxation rate is to account for smaller contribution to resistance if the scattering is happening with small change of momentum. Thus collisions not changing velocity, intra- or intervalley, do not affect the current. Explicitly performing the integration in (27), for the cases of 3D, 2D, and 1D electrons, one can make sure that, for isotropic scattering, the scattering rate coincides with the momentum relaxation rate.

Finally, we remind the reader that the transition rate must be summed over the final states within a subband, the different subbands and the equivalent valleys. The result is

$$\tau_{n,\alpha}^{-1}(k) = \sum_{n',\alpha'} \tilde{P}_{\alpha,\alpha'}(k,n,n')\,. \tag{28}$$

Mobility calculation

Our calculation of mobility is based on the linearization of the Boltzmann equation [3] for the distribution function under the influence of the electric field **F** on the particle's charge q. We assume without the loss of generality that the electric field is along the x-axis.

$$f(k) = f^0(k) + \frac{\partial f^0}{\partial E} q\tau(k) v_x(k) F\,. \tag{29}$$

The same expression applies to each subband n and each valley, α. Here the equilibrium Fermi distribution is

$$f^0(k) = \frac{1}{1+\exp\left(\dfrac{E(k)-\mu}{k_B T}\right)}. \tag{30}$$

and the particle's velocity is

$$v_x(k) = \frac{\partial E(k)}{\partial k_x}\,. \tag{31}$$

The first term in Eq. (29), the equilibrium Fermi distribution, results in zero current density. The second terms results in the current density (also along the x-axis)

$$j_{n,\alpha} = q\sum_{spin}\int f(k)v_x(k)d^D k = q^2\sum_{spin}\int \frac{\partial f^0}{\partial E}\tau_{n,\alpha}(k)v_x{}^2(k)Fd^D k\,. \tag{32}$$

The derivative of the Fermi function is $\frac{\partial f^0(E)}{\partial E} = f^0(E)\left(1-f^0(E)\right)\frac{1}{k_B T}$. And the energy over which the integration is performed is $E = \frac{Dmv_x^2}{2}$. The mobility corresponding to each subband and valley is

$$\mu_{n,\alpha} = \frac{j_{n,\alpha}}{\mathbf{F}qn_{n,\alpha}^{(D)}} = \frac{2q}{Dmn_{n,\alpha}^{(D)}k_B T}\int f^0(E)\left(1-f^0(E)\right)\tau_{n,\alpha}(E)\mathcal{D}^{(D)}(E)EdE\,. \tag{33}$$

The total mobility is obtained from the sum of all current densities in each of the subbands and valleys:

$$\mu = \sum_{n,\alpha}\frac{\mu_{n,\alpha}n_{n,\alpha}^{(D)}}{n^{(D)}}\,. \tag{34}$$

$$n^{(D)} = \sum_{n,\alpha} n_{n,\alpha}^{(D)}\,. \tag{35}$$

In order to gain physical insight into the dependencies, we focus on just one subband in case of confined structures. This is a realistic situation when the difference of energies of subband edges is much more than the thermal energy. Plugging in the relation, (26), and expressing the density in a subband via the density of states results in

$$\mu_n = \frac{\nu 2q\int f^0(E)(1-f^0(E))EdE}{\nu' Dmk_B TU\mathcal{F}_{nn}\int f^0(E)\mathcal{D}^{(D)}(E)dE}, \tag{36}$$

where ν is the number of valleys and ν' is the number of identical valleys available for states after the collision.

We consider the Boltzmann limit of the Fermi distribution $f << 1$. All these simplifying assumptions enable us to calculate mobilities in structures of various dimensionalities (where for simplicity we drop numbers of valleys) and recall that the overlap factor reduces to 1 in three dimensions:

$$\mu_1^{(3)} = \frac{(2\pi)^{3/2}q\hbar^3}{3m^{5/2}(k_B T)^{1/2}U}, \tag{37}$$

$$\mu_1^{(2)} = \frac{\pi q\hbar^2}{m^2 U\mathcal{F}_{11,z}}, \tag{38}$$

$$\mu_1^{(1)} = \frac{(2\pi)^{1/2} q\hbar (k_B T)^{1/2}}{m^{3/2} U \mathcal{F}_{11,y11,z}}. \tag{39}$$

In other words, the ratio of mobilities as one dimension becomes confined is essentially the same and is independent of the mechanism of phonon scattering

$$\boxed{\frac{\mu_1^{(2)}}{\mu_1^{(3)}} = \frac{3}{2}\sqrt{\frac{mk_B T}{2\pi}}\frac{1}{\hbar \mathcal{F}_{11,z}}}. \tag{40}$$

$$\boxed{\frac{\mu_1^{(1)}}{\mu_1^{(2)}} = 2\sqrt{\frac{mk_B T}{2\pi}}\frac{\mathcal{F}_{11,z}}{\hbar \mathcal{F}_{11,y11,z}}}. \tag{41}$$

A closed expression for the overlap factor may be obtained for a few special cases.

In the case of a square well/wire of size b with impenetrable walls, the wavefunction factorizes along each dimension to be

$$\psi_n(z) = \sqrt{\frac{2}{b}}\cos\left(\frac{\pi n z}{b}\right) \tag{42}$$

and the energies of the subband edges are

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2mb^2}. \tag{43}$$

The overlap factors are found to be

$$\mathcal{F}_{11,y11,z} = \mathcal{F}_{11,z}{}^2, \qquad \mathcal{F}_{11,z} = \frac{3}{2b}. \tag{44}$$

The condition that only one subband is filled

$$3k_B T < E_2 - E_1 \tag{45}$$

translates into the result that

$$\frac{\mu_1^{(2)}}{\mu_1^{(3)}} \le \frac{\sqrt{\pi}}{2} \qquad \frac{\mu_1^{(1)}}{\mu_1^{(2)}} \le \frac{2\sqrt{\pi}}{3} \tag{46}$$

A similar expression and a very close numerical result (0.95) is obtained with approximate expressions for wavefunctions in a triangular quantum well, which is a very good model for the inversion layer in a traditional MOS transistor or a large diameter nanowire transistor. Apparently the ratio of mobilities is less than one for most of the shapes of the confining potential.

Therefore the main result of the paper is that:

- **for one filled subband in quantum confined structures with Fermi-non-degenerate carriers, phonon-limited mobility is expected to be smaller or approximately equal to the corresponding non-confined structure.**

Without doing the detailed calculation one can still argue that the outcome is the same if multiple bands are filled. The number of channels for conduction increases but so does the number of states for scattering. The contribution of these factors cancels. For a large number of subbands the mobility goes to the limit of higher dimensionality and the above ratios must tend to unity.

Conclusion

In summary, we find that electron mobility is expected to decrease with the size as it approaches the values needed for quantum confinement. Intuitively this may be interpreted as follows. In low-dimensional structures, the density of states is decreased so that the scattering rate may decrease. However the density of states available for conduction decreases as well. Mobility is the product of the two factors, so the factor of density of states cancels out. The remaining weak dependence on geometry parameters indicates higher mobility for unconfined structures. This confirms and elucidates the conclusions of [1].

While this conclusion discourages pushing down the width of the nanowire, it does not the reason for concern about scaling down nanowire transistors. Conductance of nanowires increases in this scaling due to more ballistic transport in small nanowires.

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References

[1] R. Kotlyar, B. Obradovic, P. Matagne, M. Stettler, “An assessment of room temperature phonon mobility in gated silicon nanowires”, Appl. Phys. Lett., v. 84, p. 5270 (2004).

[2] T. Ando, A. B. Fowler, and F. Stern, “Electronic properties of two-dimensional systems”, Rev. of Mod. Phys., v. 54, no. 2, pp. 437-672, April 1982.

[3] P. Y. Yu and M. Cardona, *Fundamentals of Semiconductors* (Springer, Berlin, 1996).

[4] C. Jacoboni and L. Reggiani, “The Monte Carlo method for the solution of charge transport in semiconductors with applications to covalent materials”, Rev. of Mod. Phys., v. 55, no. 3, pp. 645-705, July 1983.

Notes on the mobility in 1D, 2D, and 3D Systems

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Transition probability:

$$S^{(\pm)}(\mathbf{k}',\mathbf{k}) = \frac{2\pi}{\hbar}\left|\langle \mathbf{k}',c'|U_S|\mathbf{k},c\rangle\right|^2 \delta\left(E(k',c') - E(k,c)\right) \tag{1.1}$$

Scattering Potential:

$$U_S = \sum_{\mathbf{q}} C(q) u_{\mathbf{q}} = \sum_{\mathbf{q}} C(q)\sqrt{\frac{\hbar}{2mN\omega(q)}}\left(a_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} + a^{\dagger}_{-\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{r}}\right)$$

$$U_S^{+} = \sum_{\mathbf{q}} C(q)\sqrt{\frac{\hbar}{2mN\omega(q)}} a_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}}$$

$$U_S^{-} = \sum_{\mathbf{q}} C(q)\sqrt{\frac{\hbar}{2mN\omega(q)}} a^{\dagger}_{-\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{r}}$$

$$\langle c'|U_S^{\pm}|c\rangle = \sum_{\mathbf{q}} C(q)\sqrt{\frac{\hbar}{2mN\omega(q)}}\left(n_q + \frac{1}{2} \mp \frac{1}{2}\right) e^{\pm i\mathbf{q}\cdot\mathbf{r}} \tag{1.2}$$

Now, for AP using Debye spectrum and equipartition of energy and for OP using Einstein model, we get:

For AP:

$$C(q) = D_{ac} q$$

$$\left(n_q + \frac{1}{2} - \frac{1}{2} \right) \approx \left(n_q + \frac{1}{2} + \frac{1}{2} \right) \approx \frac{k_B T}{\hbar \omega(q)}$$

$$\omega(q) = u_l q \tag{1.3}$$

For OP:

$$C(q) = D_{op}$$

$$\omega(q) = \omega_o$$

$$n_q = n_o$$

Using these into Eq. (1.3) we get:

$$\left\langle c' \left| U_S^{\pm} \right| c \right\rangle = \left\{ \begin{array}{l} \frac{1}{\sqrt{\Omega}} \sqrt{\frac{D_{ac}^2 k_B T}{2 \rho u_l^2}} \\ \frac{1}{\sqrt{\Omega}} \sqrt{\frac{D_{op}^2 \hbar}{2 \rho \omega_o} \left(n_q + \frac{1}{2} \mp \frac{1}{2} \right)} \end{array} \right\} e^{\pm i \mathbf{q} \cdot \mathbf{r}} = \sqrt{\frac{U_{ac}^{op}}{\Omega}} e^{\pm i \mathbf{q} \cdot \mathbf{r}} \tag{1.4}$$

Using these into Eq. (1.1) we get:

$$\begin{aligned} S^{(\pm)}(\mathbf{k}, \mathbf{k}') &= \frac{2\pi}{\hbar} \frac{U}{\Omega} \sum_{\mathbf{q}} \left| \left\langle \mathbf{k}' \left| e^{\pm i \mathbf{q} \cdot \mathbf{r}} \right| \mathbf{k} \right\rangle \right|^2 \delta\left(E(k') - E(k) \mp \hbar \omega_o \right) \\ &= \frac{2\pi}{\hbar} \frac{U}{\Omega} \delta\left(E(k') - E(k) \mp \hbar \omega_o \right) \sum_{\mathbf{q}} \left| \left\langle \mathbf{k}' \left| e^{\pm i \mathbf{q} \cdot \mathbf{r}} \right| \mathbf{k} \right\rangle \right|^2 \\ &= \frac{2\pi}{\hbar} \frac{U}{\Omega} \delta\left(E(k') - E(k) \mp \hbar \omega_o \right) \sum_{\mathbf{q}} \delta_{\mathbf{k}'_{uc}, \mathbf{k}_{uc} \pm \mathbf{q}_{uc}} \left| F_{n,n}(\mathbf{q}_{cf}) \right|^2 \\ &= \frac{2\pi}{\hbar} \frac{U}{\Omega} \delta\left(E(k') - E(k) \mp \hbar \omega_o \right) \times \delta_{\mathbf{k}'_{uc}, \mathbf{k}_{uc} \pm \mathbf{q}_{uc}} \left(\frac{3}{2} \right)^{3-d} \end{aligned} \tag{1.5}.$$

Scattering rate:

$$\frac{1}{\tau(\mathbf{k})} = \sum_{\mathbf{k}'} S^{(\pm)}(\mathbf{k},\mathbf{k}')$$
$$= U\left(\frac{3}{2}\right)^{3-d} \times \frac{1}{\Omega}\sum_{\mathbf{k}'} \delta_{\mathbf{k}'_{uc},\mathbf{k}_{uc}\pm\mathbf{q}_{uc}} \times \delta\left(E(k') - E(k) \mp \hbar\omega_o\right)$$
$$= U\left(\frac{3}{2}\right)^{3-d} \times \frac{1}{\Omega}\sum_{\mathbf{q}_{uc}} \delta\left(E(k'_{uc}) - E(k_{uc}) \mp \hbar\omega_o\right) \qquad (1.6).$$

Eq. (1.6) *tells that scattering rate is proportional to : i) form factor* $(3/2)^{3-d}$ *and ii) on the* ***ratio*** *of electronic DOS to the # of harmonic oscillators.*

1. *We have seen that the form factor tend to increase the scattering rate.*

2. *In the following we will see how ratio of DOS to the # of oscillators scales with dimensionality.*

For different dimensions:

$$\frac{1}{\tau^{(3d)}(\mathbf{k})} = U\frac{1}{\Omega}\frac{\Omega}{(2\pi)^3}\frac{4\pi k'^2_{uc}}{\nabla_k E(k'_{uc})}$$
$$\frac{1}{\tau^{(2d)}(\mathbf{k})} = U\left(\frac{3}{2}\right)\frac{1}{\Omega}\frac{A}{(2\pi)^2}\frac{2\pi k'_{uc}}{\nabla_k E(k'_{uc})} \qquad (1.7).$$
$$\frac{1}{\tau^{(1d)}(\mathbf{k})} = U\left(\frac{3}{2}\right)^2\frac{1}{\Omega}\frac{L}{2\pi}\frac{1}{\nabla_k E(k'_{uc})}$$

After evaluating the gradient of the E-k we get,

$$E = \frac{\hbar^2 k^2}{2m^*}$$
$$\nabla_k E(k) = \frac{\hbar^2}{m^*}k \qquad (1.8)$$

Define:

$$\beta \equiv \frac{\sqrt{2m^* k_B T}}{\hbar} \tag{1.9}$$

$$\eta \equiv \left(\frac{\hbar^2 k^2}{2m^*}\right) \Big/ k_B T = \left(\frac{k}{\beta}\right)^2$$
$$\Rightarrow k = \beta\sqrt{\eta} \tag{1.10}$$
$$\Rightarrow dk = \frac{\beta}{2}\eta^{-1/2}$$

Putting these into Eq. (1.7) we get,

$$\frac{1}{\tau^{(3d)}(\mathbf{k})} = U\frac{1}{\Omega}\frac{\Omega}{(2\pi)^3}\frac{4\pi k'^2_{uc}}{\frac{\hbar^2}{m^*}k'_{uc}} = \left(\frac{m^* U\beta}{2\pi\hbar^2}\right)\frac{1}{\pi}\eta^{1/2}$$
$$\frac{1}{\tau^{(2d)}(\mathbf{k})} = U\left(\frac{3}{2}\right)\frac{1}{\Omega}\frac{A}{(2\pi)^2}\frac{2\pi k'_{uc}}{\frac{\hbar^2}{m^*}k'_{uc}} = \left(\frac{m^* U\beta}{2\pi\hbar^2}\right)\left(\frac{3}{2\beta t}\right)\eta^0 \tag{1.11}.$$
$$\frac{1}{\tau^{(1d)}(\mathbf{k})} = U\left(\frac{3}{2}\right)^2\frac{1}{\Omega}\frac{L}{(2\pi)}\frac{1}{\frac{\hbar^2}{m^*}k'_{uc}} = \left(\frac{m^* U\beta}{2\pi\hbar^2}\right)\left(\frac{3}{2\beta t}\right)^2\eta^{-1/2}$$

Scattering time:

$$\tau^{(3d)}(\mathbf{k}) = \left(\frac{2\pi\hbar^2}{m^* U\beta}\right)\pi\eta^{-1/2}$$
$$\tau^{(2d)}(\mathbf{k}) = \left(\frac{2\pi\hbar^2}{m^* U\beta}\right)\left(\frac{2\beta t}{3}\right)\eta^0 \tag{1.12}.$$
$$\tau^{(1d)}(\mathbf{k}) = \left(\frac{2\pi\hbar^2}{m^* U\beta}\right)\left(\frac{2\beta t}{3}\right)^2\eta^{1/2}$$

Taking the ratios we find:

$$\frac{\tau^{(2d)}}{\tau^{(3d)}} = \frac{2}{3}\left(\frac{\beta t}{\pi}\right)\sqrt{\eta}$$
$$\frac{\tau^{(1d)}}{\tau^{(2d)}} = \frac{2}{3}(\beta t)\sqrt{\eta} \qquad (1.13).$$

*Thus for scattering **time** the ratio of electronic DOS to # of oscillators scales as $(\beta t/\pi)$ from 3D to 2D, and as (βt) from 2D to 1D. For strongly confined regime with 1 subband transport:*

$$\beta t \le \pi \qquad (1.14).$$

Putting the above we get:

$$\frac{\tau^{(2d)}}{\tau^{(3d)}} \le \frac{2}{3}\sqrt{\eta}$$
$$\frac{\tau^{(1d)}}{\tau^{(2d)}} \le \frac{2\pi}{3}\sqrt{\eta} \qquad (1.15)$$

1. *Thus we see a degradation (low scattering time) of scattering time pre-factor going from 3D to 2D, and*
2. *an enhancement (high scattering time) going from 2D to 1D.*

Mobility Calculation:

For any scattering rate of the form, $\boxed{\tau(\mathbf{k}) = \tau_{do}\eta^{s}}$, and with dimensionality d, the mobility in the non-degenerate limit can be written as:

$$\mu^{(d)} = \frac{q}{m^*}\tau_{do}\frac{\Gamma\left(s + d/2 + 1\right)}{\Gamma\left(d/2 + 1\right)} \qquad (1.16).$$

For the cases we are considering; s can be expressed as: $\boxed{s = 1 - \frac{d}{2}}$ and we get:

$$\boxed{\mu^{(d)} = \frac{q\tau_{do}}{m^*}\frac{\Gamma(2)}{\Gamma\left(d/2+1\right)}} \qquad (1.17).$$

$$\begin{aligned}
\mu^{(3)} &= \frac{q}{m^*}\left(\frac{2\pi\hbar^2}{m^*U\beta}\right)\frac{4\sqrt{\pi}}{3} \\
\mu^{(2)} &\le \frac{q}{m^*}\left(\frac{2\pi\hbar^2}{m^*U\beta}\right)\frac{2\pi}{3} \\
\mu^{(1)} &\le \frac{q}{m^*}\left(\frac{2\pi\hbar^2}{m^*U\beta}\right)\left(\frac{2\pi}{3}\right)^2\frac{2}{\sqrt{\pi}}
\end{aligned} \qquad (1.18)$$

$$\begin{aligned}
\frac{\mu^{(2)}}{\mu^{(3)}} &\le \frac{\sqrt{\pi}}{2} \\
\frac{\mu^{(1)}}{\mu^{(2)}} &\le \frac{4\sqrt{\pi}}{3}
\end{aligned} \qquad (1.19)$$

Thus in the 1D case mobility enhancement of $\frac{4\sqrt{\pi}}{3}$ is observed.