Open 1D Systems:
Transmission through & over 1 Barrier

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Scattering Matrix approach

Define our system: Single barrier

One matrix each for each interface: 2 S-matrices

Incident: A
Reflected: B
Transmitted: E
Incident: F

No particles lost! Typically A=1 and F=0.
Tunneling through a single barrier

Wave-function each region,

\[ \psi_1(x) = Ae^{ikx} + Be^{-ikx} \]
\[ \psi_2(x) = Ce^{-\gamma x} + De^{\gamma x} \]
\[ \psi_3(x) = Ee^{ikx} + Fe^{-ikx} \]

\[ k = \sqrt{\frac{2mE}{\hbar^2}} \]
\[ \gamma = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} \]
Applying boundary conditions at each interface (x=0 and x=L) gives,

\[ \psi_1(0) = \psi_2(0) \quad \Rightarrow \quad A + B = C + D \]
\[ \psi_1'(0) = \psi_2'(0) \quad \Rightarrow \quad ik(A - B) = -\gamma(C - D) \]
\[ \psi_2(L) = \psi_3(L) \quad \Rightarrow \quad Ce^{-\gamma L} + De^{\gamma L} = Ee^{ikL} + Fe^{-ikL} \]
\[ \psi_2'(L) = \psi_3'(L) \quad \Rightarrow \quad -\gamma \left( Ce^{-\gamma L} - De^{\gamma L} \right) = ik \left( Ee^{ikL} - Fe^{-ikL} \right) \]
Tunneling through a single barrier

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\[ k = \sqrt{\frac{2mE}{\hbar^2}} \quad \gamma = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} \]
Applying boundary conditions at each interface (x=0 and x=L) gives,

\[ \psi_1(0) = \psi_2(0) \implies A + B = C + D \]

\[ \psi_1(0) = \psi_2(0) \implies ik(A - B) = -\gamma(C - D) \]

\[ \psi_2(L) = \psi_3(L) \implies Ce^{-\gamma L} + De^{\gamma L} = Ee^{ikL} + Fe^{-ikL} \]

\[ \psi_2(L) = \psi_3(L) \implies -\gamma(CE^{-\gamma L} - De^{\gamma L}) = ik(EE^{ikL} - Fe^{-ikL}) \]

Which in matrix can be written as,

\[
\begin{bmatrix}
A \\
B
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} \left( 1 + i \frac{\gamma}{k} \right) & \frac{1}{2} \left( 1 - i \frac{\gamma}{k} \right) \\
\frac{1}{2} \left( 1 - i \frac{\gamma}{k} \right) & \frac{1}{2} \left( 1 + i \frac{\gamma}{k} \right)
\end{bmatrix} \begin{bmatrix}
C \\
D
\end{bmatrix} = M_1 \begin{bmatrix}
C \\
D
\end{bmatrix}
\]

\[
\begin{bmatrix}
C \\
D
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} \left( 1 - i \frac{k}{\gamma} \right) e^{(ik + \gamma)L} & \frac{1}{2} \left( 1 + i \frac{k}{\gamma} \right) e^{-(ik - \gamma)L} \\
\frac{1}{2} \left( 1 + i \frac{k}{\gamma} \right) e^{(ik - \gamma)L} & \frac{1}{2} \left( 1 - i \frac{k}{\gamma} \right) e^{-(ik + \gamma)L}
\end{bmatrix} \begin{bmatrix}
E \\
F
\end{bmatrix} = M_2 \begin{bmatrix}
E \\
F
\end{bmatrix}
\]
Transmission can be found using the relations between unknown constants,

\[ T(E) = \left| \frac{E}{A} \right|^2 = \frac{1}{|m_{11}|^2} \]

\[ \begin{bmatrix} A \\ B \end{bmatrix} = M_1 \begin{bmatrix} C \\ D \end{bmatrix} = M_1 M_2 \begin{bmatrix} E \\ F \end{bmatrix} = M \begin{bmatrix} E \\ F \end{bmatrix} \]

**Case: \( E < V_o \)**

\[ T(E) = \left[ 1 + \left( \frac{\gamma^2 + k^2}{2k\gamma} \right)^2 \right]^{-1} \]

**Case (\( \gamma L \) large): Strong barrier**

\[ T(E) \approx \left( \frac{4k\gamma}{k^2 + \gamma^2} \right)^2 \exp(-2\gamma L) \]

**Case (\( \gamma L \ll 1 \)): Weak barrier**

\[ T(E) \approx \frac{1}{1 + \left( kL / 2 \right)^2} \]

**Case: \( E > V_o \)**

\[ T(E) = \left[ 1 + \left( \frac{k^2 - k_2^2}{2kk_2} \right)^2 \sin^2(k_2L) \right]^{-1} \]
• Transmission is finite under the barrier – tunneling!
• Transmission above the barrier is not perfect unity!
• Quasi-bound state above the barrier.
  Transmission goes to one.
• Increased barrier width reduces tunneling probability
• Thicker barrier increase the reflection probability below the barrier height.
• Quasi-bound states occur for the thicker barrier too.
Effect of barrier thickness above the barrier

- Increased barrier width increases oscillation frequency in transmission and reflection.

- Quasi-bound states above the barrier due to 2 reflections.
Increasing the barrier height does not have a significant effect on the modulation frequency above the barrier height.

Oscillations are strongly related to barrier width but not height!
• $E < V_0$: Classical Physics: $T(E) = 0$,
  Quantum Physics: a hyperbolic increase.

• $E > V_0$: Classical Physics: $T(E) = 1$,
  Quantum Physics: total transmission at discrete energies only.
  Only barriers of certain width will transmit all particles at a given energies.
• Quantum wavefunctions can tunnel through barriers
• Tunneling is reduced with increasing barrier height and width
• Transmission above the barrier is not unity
  » 2 interfaces cause constructive and destructive interference
  » Quasi bound states are formed that result in unity transmission