Lecture 14
Cantilever eigenmodes, equivalent point mass oscillator

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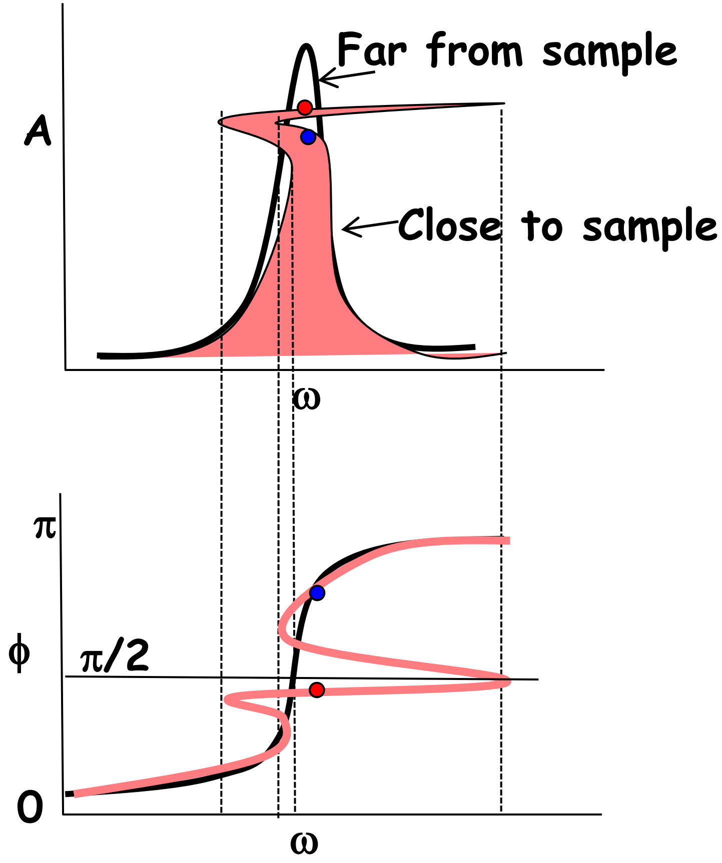
Attractive vs. repulsive mode imaging



- Depending on operating conditions, its is possible to app roach in attractive regime and transition to repulsive at some Z

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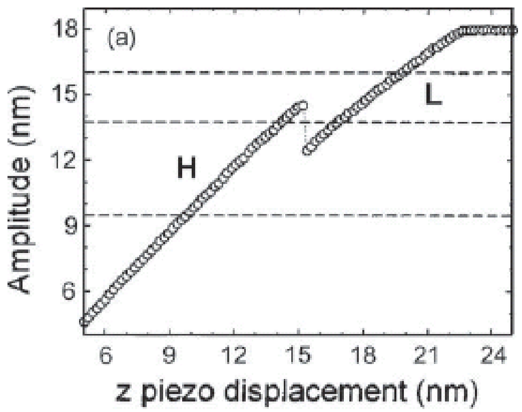
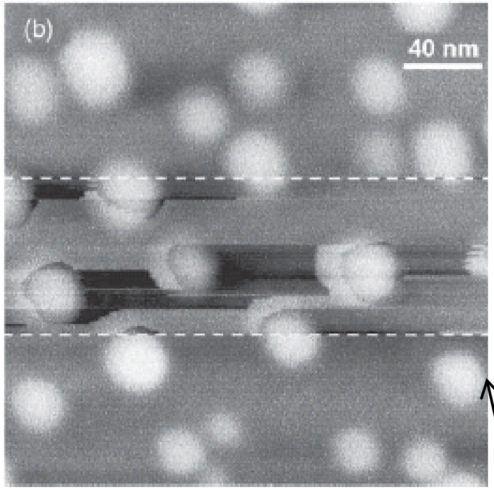
Implications





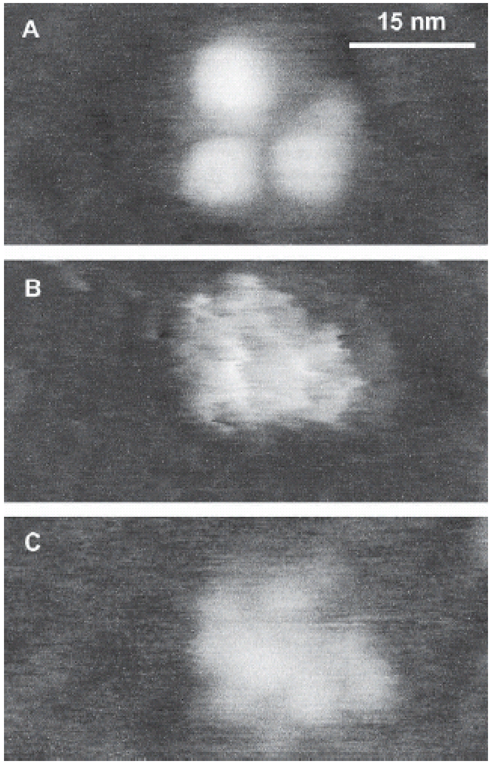
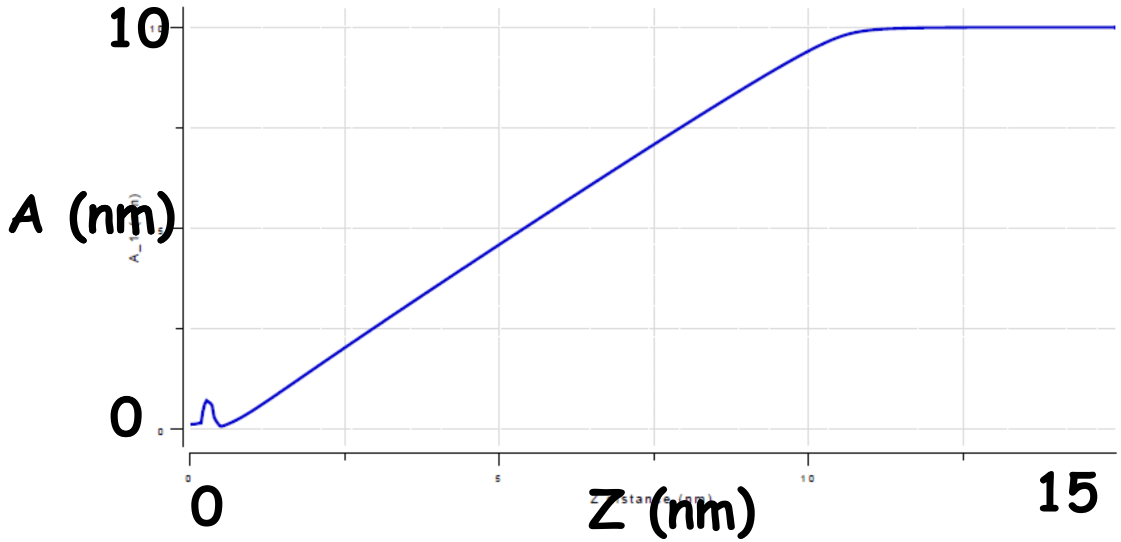


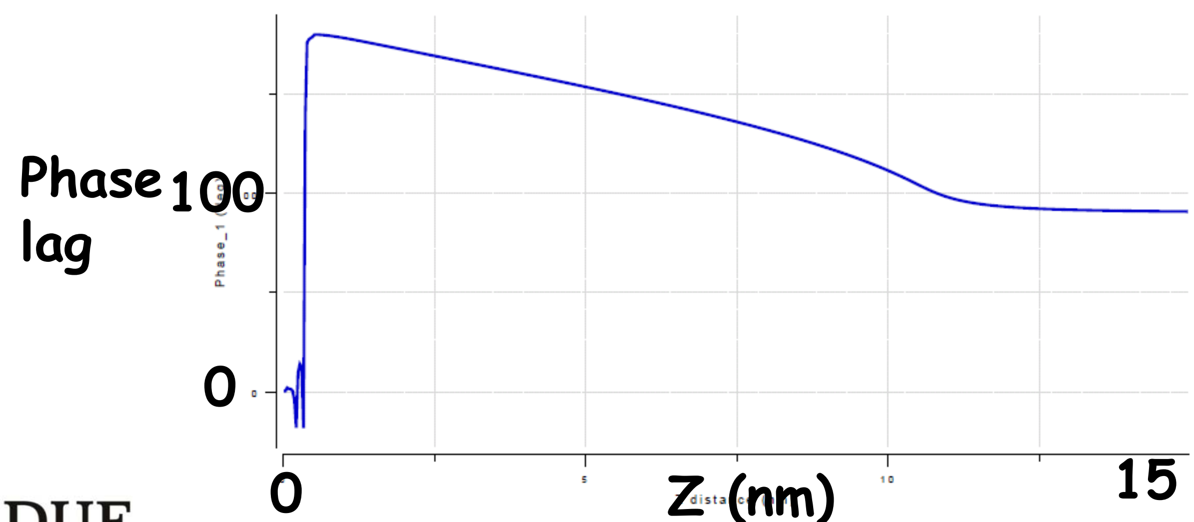
Fig. 11. Experimental determination of the low and high amplitude branches. (a) Amplitude curve, the L and H branches are plotted by open circles. Dashed lines indicate the A_{sp} values used to image a 200 × 200 nm^2 InAs quantum dot sample. (b) The system evolves from stable imaging in the L state A_{sp} = 16 nm (top) to unstable imaging due to switching between H and L states A_{sp} = 13.8 nm (middle) and finally to stable imaging in the H state A_{sp} = 9.5 nm (bottom). Adapted from [56].

Fig. 12. (A) High-resolution image of a single a-HSA (obtained by operating in an L state). The three fragments and the hinge regions are clearly resolved. (B) Image of the same molecule obtained by operating the instrument in an H state. (C) Image of the molecule in the initial L state after repeated imaging in an H state. The characteristic shape of the molecule has been lost by imaging in an H state. Adapted from [7].

Factors controlling this phenomenon

- Rtip=10nm,k=40 N/m, Ao=10nm, Q=500, $f=f_0$=300 kHz, H=2*E-19, Fad=1.4nN, E*=1 GPa





Factors controlling this phenomenon

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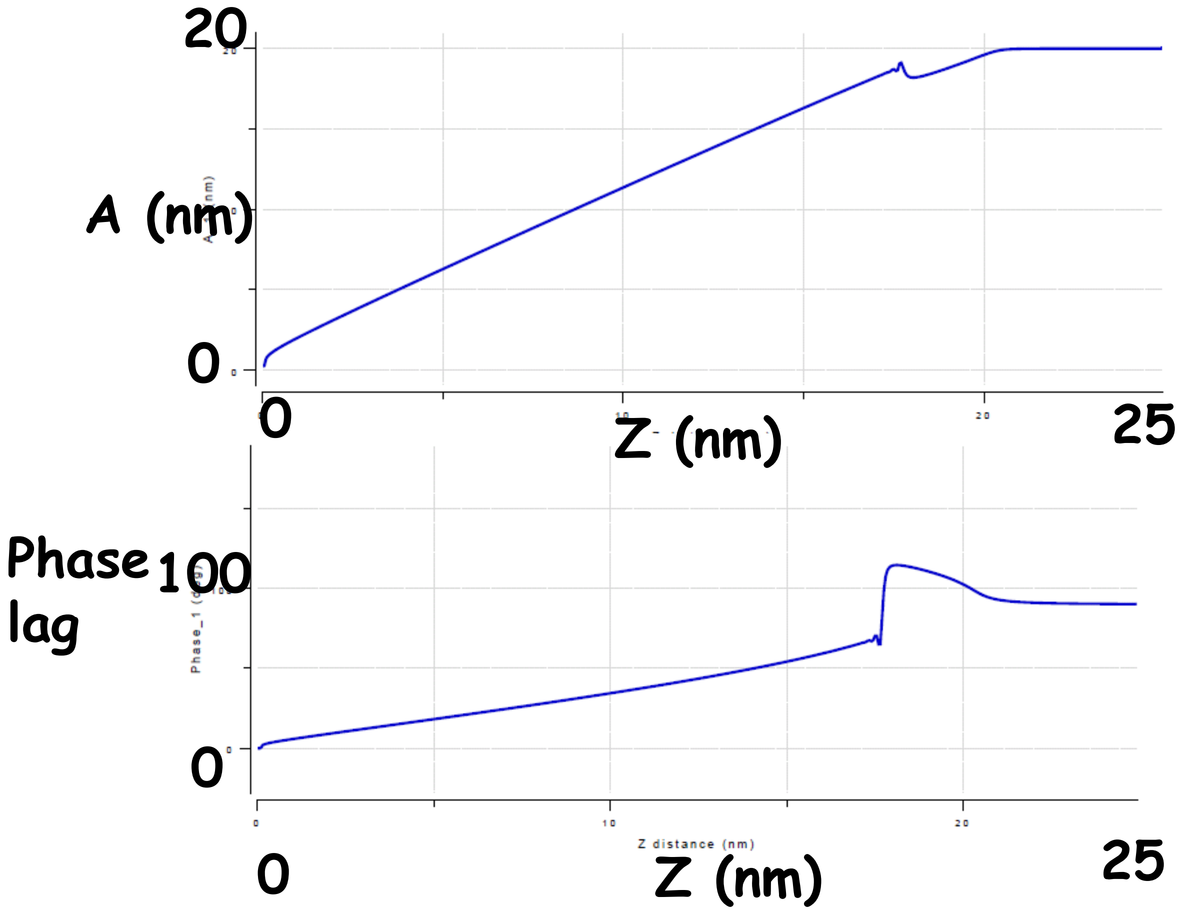

- Increase amplitude to tap (repulsive regime)

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Factors controlling this phenomenon

- Rtip=10nm,k=40 N/m, Ao=20nm, Q=700, f=299kHz, H=

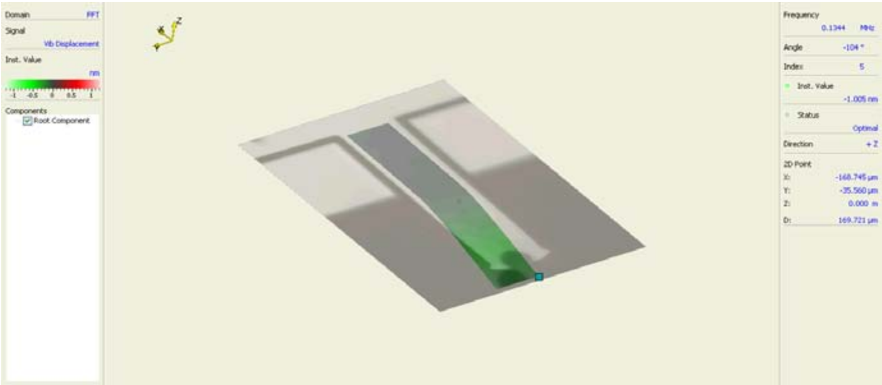
A (

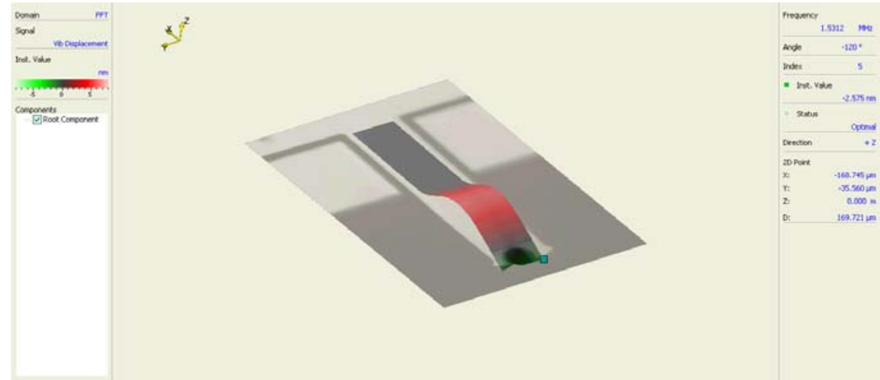
Phase lag 1

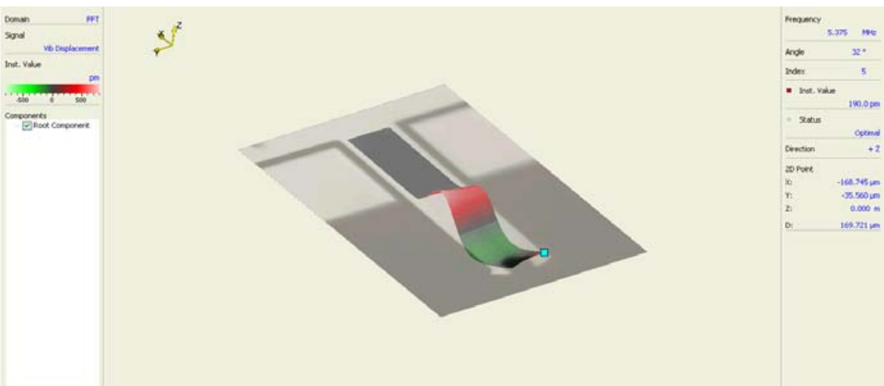
0 z (nm)

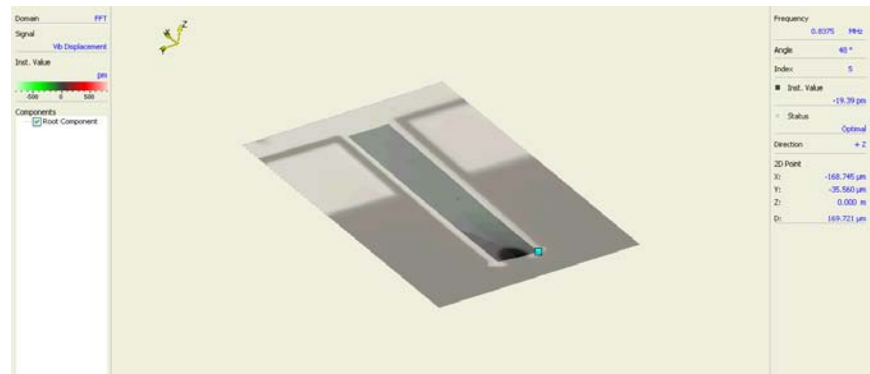
- Increase Q to stay in attractive regime

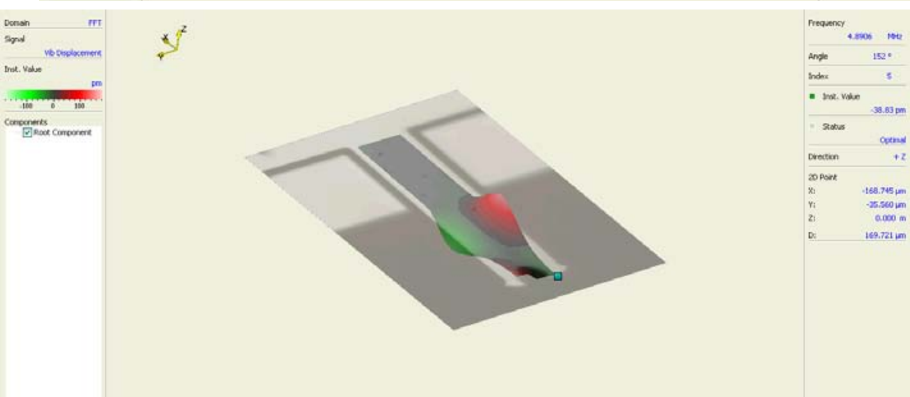
Cantilever eigenmodes

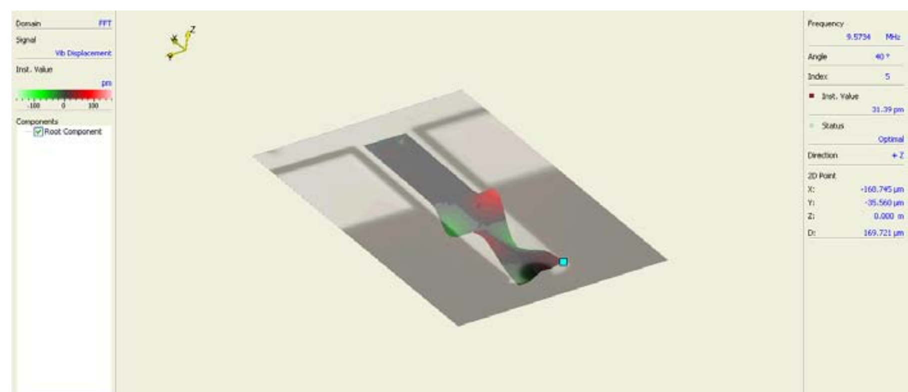








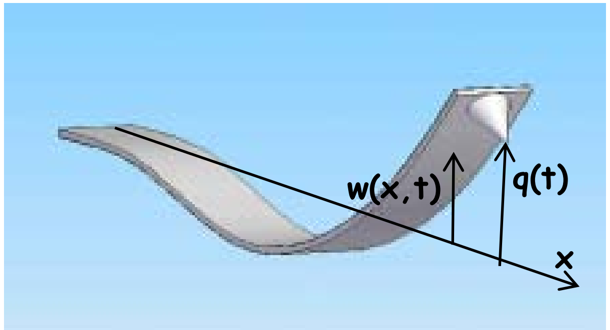


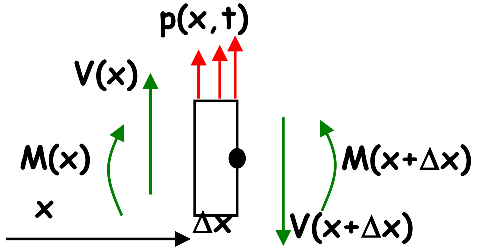


Point mass vs. continuous oscillator?

- The point mass model was derived with the assumption that cantilever mass was « tip mass
- The shape of the oscillating beam in the point mass model is assumed to be that of a statically bent beam under a tip force
- The point mass model does not predict any oscillation modes beyond the fundamental
- How to include spatially continuous nature of the AFM cantilever and yet enjoy the simplicity of a point mass model?

Transverse vibrations of classical beam





p(x,t): external force per unit length

A: Area of cross section

ρ: mass density of cantilever

- Bernoulli-Euler beam theory

$$V(x) + p(x,t)\Delta x - V(x+\Delta x) = (\rho A \Delta x)\ddot{w},$$

$$\text{as } \Delta x \to 0 \text{ we get } \rho A \ddot{w} = -\frac{\partial V}{\partial x} + p(x,t)$$

or

$$\rho A \ddot{w} = -\frac{\partial^2 M}{\partial x^2} + p(x,t) = -EI\frac{\partial^4 w}{\partial x^4} + p(x,t)$$

Or

$$\rho A \ddot{w} + EI\frac{\partial^4 w}{\partial x^4} = p(x,t)$$

To be solved with boundary conditions

$$w(0) = 0$$

$$\frac{\partial w}{\partial x}(0) = 0$$

$$V(L) = EI\frac{\partial^3 w}{\partial x^3}(L) = ??$$

$$M(L) = EI\frac{\partial^2 w}{\partial x^2}(L) = ??$$

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Transverse vibrations of classical beam

- To calculate eigenmodes and natural frequencies, one can set p(x,t)=0 and any damping=0

$$\rho A\ddot{w} + EI\frac{\partial^4 w}{\partial x^4} = 0 \qquad (1)$$

$$Let \quad w(x,t) = \phi(x)T(t) \qquad (2)$$

$$\frac{\left(EI/\rho A\right)}{\phi(x)}\frac{d^4\phi(x)}{dx^4} = -\frac{1}{T(t)}\frac{d^2T}{dt^2} = const = \omega^2 \qquad (3)$$

$$T(t) = A\sin(\omega t) + B\cos(\omega t) \qquad \phi(x) = Ce^{\lambda x} \qquad (4)$$

$$(4)\ in\ (3a) \rightarrow$$

$$\lambda^4 = \beta^4 = \frac{\rho A\omega^2}{EI} \Rightarrow \lambda_{1,2} = \pm\beta,\ \lambda_{3,4} = \pm i\beta \qquad (5)$$

$$\phi(x) = C_1e^{\beta x} + C_2e^{-\beta x} + C_3e^{i\beta x} + C_4e^{-i\beta x}$$

$$\cong C_1\sin(\beta x) + C_2\cos(\beta x) + C_3\sinh(\beta x) + C_4\cosh(\beta x)$$

$$and \quad \omega^2 = \beta^2\sqrt{\frac{EI}{\rho A}}$$

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Transverse vibrations of classical beam

- Assuming negligible tip mass

$$\phi(x) = C_1\cos(\beta x) + C_2\sin(\beta x) + C_3\cosh(\beta x) + C_4\sinh(\beta x) \; where \; \beta^4 = \rho A\omega^2 / EI \qquad (1)$$

$$w(0) = 0\,, \quad \frac{\partial w}{\partial x}(0) = 0, \quad EI\frac{\partial^3 w}{\partial x^3}(L) = 0, \quad EI\frac{\partial^2 w}{\partial x^2}(L) = 0$$

$$C_1 = C_3 = 0 \; and$$

$$C_2(\cos(\beta L) + \cosh(\beta L)) + C_4(\sin(\beta L) + \sinh(\beta L)) = 0 \qquad (2)$$

$$C_2(-\sin(\beta L) + \sinh(\beta L)) + C_4(\cos(\beta L) + \cosh(\beta L)) = 0$$

$$or \begin{bmatrix} \cos(\beta L) + \cosh(\beta L) & \sin(\beta L) + \sinh(\beta L) \\ -\sin(\beta L) + \sinh(\beta L)) & \cos(\beta L) + \cosh(\beta L) \end{bmatrix} \begin{bmatrix} C_2 \\ C_4 \end{bmatrix} \qquad (3)$$

$$for \; solutions \; where \; C_2, C_4 \neq 0 \; we \; must \; have$$

$$\cos(\beta L)\cosh(\beta L) + 1 = 0 \qquad (4)$$

$$and \quad C_4 = -\frac{\cos(\beta L) + \cosh(\beta L)}{\sin(\beta L) + \sinh(\beta L)}C_2 \qquad (5)$$

$$Solving \; (4) \;\; yields \; (\beta L)_1 = 1.875, (\beta L)_2 = 4.694, (\beta L)_3 = 7.855 \ldots\ldots$$

So that the eigen modes are

$$\phi_n(x) = (\cos(\beta_n x) - \cosh(\beta_n x)) - \frac{\cos(\beta_n L) + \cosh(\beta_n L)}{\sin(\beta_n L) + \sinh(\beta_n L)}\left(\sin(\beta_n x) - \sinh(\beta_n x)\right) \; (6)$$

$$Normalize \; so \; that \; \phi_n(L) = 1$$

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Eigenmodes

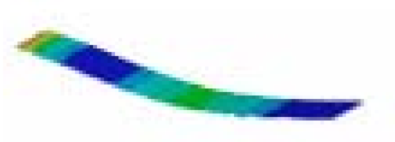
1st eigenmode

$$(\beta L)_1 = 1.875,\ (\beta L)_2 = 4.694,\ (\beta L)_3 = 7.855\ldots\ldots$$

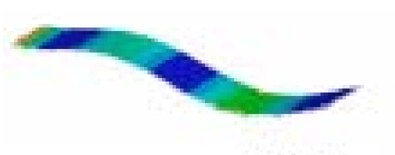
Thus

$$\omega_1 : \omega_2 : \omega_3 : .... = 1 : 6.26 : 17.55 : ...$$

for a uniform rectangular lever with negligible tip mass



2nd eigenmode



3rd eigenmode

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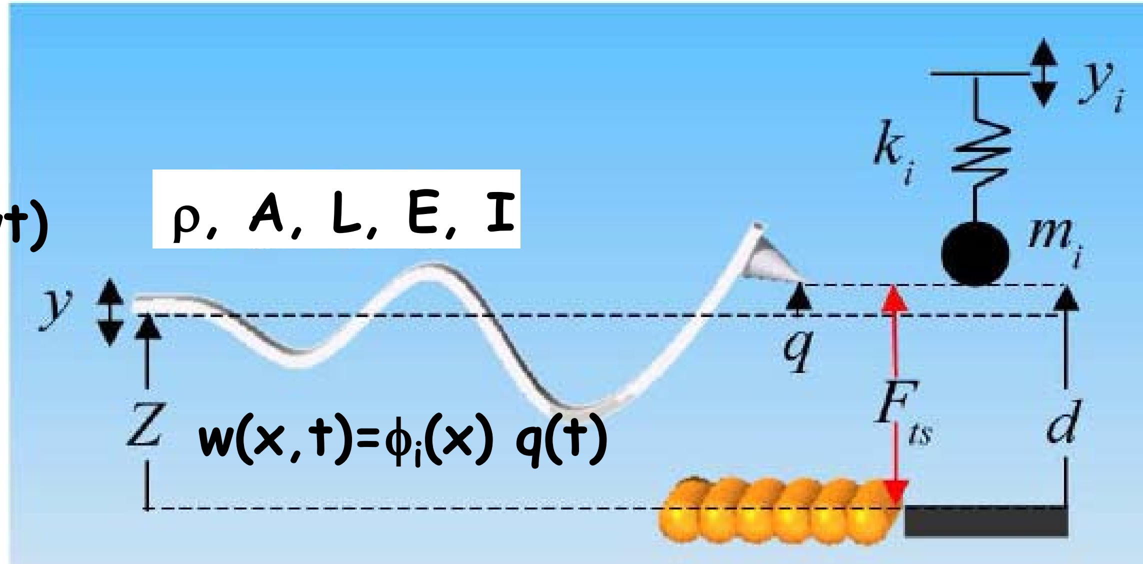
Equivalent point mass oscillator

$y(t)=Y\cos(wt)$



$y_i(t)=Y_i\cos(wt)$

- Energy based equivalence principle

$$\frac{1}{2}k_i q^2 = \frac{1}{2}\int_{x=0}^{x=L} EI\left(\frac{d^2\phi_i}{dx^2}\right)^2 dx \quad \frac{1}{2}m_i q^2 = \frac{1}{2}\int_{x=0}^{x=L} \rho A\left(\phi_i\right)^2 dx \quad Y_i = Y\left(\frac{\omega}{\omega_i}\right)^2 \frac{\int_0^L \phi_i(x)dx}{\int_0^L \left(d^2\phi_i / dx^2\right)dx}$$

For negligible tip mass

$$k_1 = 1.03k,\ k_2 = 40.5k,\ k_3 = 317k$$

$$m_1 = m_2 = m_3 = ... = 0.249\ \rho AL$$

$$Y_1 \sim 1.5Y$$

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Melcher *et al*, App. Phys. Lett. 91(5), 2007