Derivation of Planck's Law

This page provides a brief derivation of Planck’s law from basic statistical principles. For more information, the reader is referred to the textbook by Rybicki and Lightman (*Radiative Processes in Astrophysics*, Wiley, 2004) http://books.google.com/books?id=LtdEjNABMlsC&dq=isbn:0471827592&ei=0KPFSOKvE4mIjwGQ2Oj3BA. The reader might also find interest in the historical development of early research in radiation physics as surveyed by Barr http://scitation.aip.org/getabs/servlet/GetabsServlet?prog=normal&id=AJPIAS000028000001000042000001&idtype=cvips&gifs=yes.

Photon gas in a box

First, consider a cubic box with each side of length *L* that is filled with electromagnetic (EM) radiation (a so-called ‘photon gas’) that forms standing waves whose allowable wavelengths are restricted by the size of the box. We will assume that the waves do not interact and therefore can be separated into the three orthogonal Cartesian directions such that the allowable wavelengths are:

$$\lambda_i = \frac{2L}{n_i} $$

where n_i is an integer greater than zero, and *i* represents one of the three Cartesian directions—*x, y,* or *z*.

From quantum mechanics, the energy of a given mode (*i.e.,* an allowable set n_x, n_y, n_z) can be expressed as

$$E(N) = \left(N + \frac{1}{2} \right) \frac{hc}{2L} \sqrt{n_x^2 + n_y^2 + n_z^2}$$

where *h* is Planck’s constant (6.626×10^{-34}J s). The number *N* represents the number of such modes, or photons, of the given energy. Importantly, unlike electrons, an unlimited number of modes, or photons, of a given energy can exist; thus, photons are governed by Bose-Einstein statistics.

Statistical mechanics of the photon gas

To derive the energy density in this photon gas, we first need to know the relative probability with which a given energy state $E(N)$ is occupied at a given temperature. Here, we turn to statistical mechanics, which reveals this probability as

$$P_N = \frac{\exp (-\beta E(N))}{Z(\beta)}$$

where β is the inverse of thermal energy, or $\beta = (k_B T)^{-1}$, and $Z(\beta)$ is a factor, called the partition function, that normalizes the probability as

$$Z(\beta) = \sum_{N=0}^{\infty} \exp (-\beta E(N)) = \frac{1}{1-\exp (-\beta \varepsilon)}$$

where $$\varepsilon = \frac{hc}{2L} \sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{hc}{\lambda} $$ is the energy of a single photon, and the latter equality derives from the relationship between the wavelength $$\lambda$$ and the $$n_i$$ indices of the EM waves in the box. This wavelength is related to the speed of light *c* and frequency $$\nu$$ through the familiar relation

$$\frac{c}{\lambda} = \nu \Rightarrow \epsilon = h \nu $$

Again from statistical mechanics (and specifically Bose-Einstein statistics), the average energy within a given mode (which is related to the average number of photons *N*) can be expressed as

$$ \langle E(N) \rangle = - \frac{d \ln Z}{d \beta} = \frac{\varepsilon}{\exp (\beta \varepsilon) - 1}$$

Energy density of the photon gas

Now that we have an expression for the average energy of a given mode, we can sum (integrate) over all modes to find the total energy within the photon gas. The total energy can be expressed as an integral over all energies as

$$U = \int_0^\infty \langle E \rangle g(\varepsilon) d\varepsilon = \int_0^\infty \frac{\varepsilon}{\exp (\beta \varepsilon) - 1} g(\varepsilon) d\varepsilon$$

where $$g(\varepsilon)$$ is an important function called the density of states. This function gives the number of allowed modes per unit energy within an interval between $$\varepsilon$$ and $$\varepsilon + d \varepsilon$$. This function can be derived from the allowable wavelengths and ‘n’ indices as

$$ g(\varepsilon) d \varepsilon = \frac{8\pi L^3}{h^3 c^3} \varepsilon^2 d \varepsilon $$

Thus, the energy per unit volume can be expressed as

$$\frac{U}{L^3} = \int_0^\infty \frac{8\pi}{h^3 c^3}\frac{\varepsilon^3}{\exp (\beta \varepsilon) - 1} d\varepsilon$$

where the integrand is the spectral energy density $$u$$. This function can be expressed in terms of energy, wavelength, or frequency through the relation $$ \varepsilon = hc/\lambda $$ such that different forms of *u* are commonly used. However, they are each integrands in expressions that are used to calculate the overall energy density as

$$\frac{U}{L^3} = \int_0^\infty u(\varepsilon, T) d\varepsilon = \int_0^\infty u(\lambda, T) d\lambda = \int_0^\infty u(\nu, T) d\nu$$

The corresponding expressions for spectral energy density follow:

$$ u(\varepsilon, T) = \frac{8\pi}{h^3 c^3}\frac{\varepsilon^3}{\exp \left(\frac{\varepsilon}{k_B T} \right) - 1} $$

$$ u(\lambda, T) = \frac{8\pi h c}{\lambda^5} \frac{1}{\exp \left(\frac{hc}{\lambda k_B T} \right) - 1}$$

$$ u(\nu, T) = \frac{8\pi h \nu^3}{c^3} \frac{1}{\exp \left(\frac{h\nu}{k_B T} \right) - 1}$$

Blackbody emission intensity

Now assume that a small hole is cut into the box. All radiation emanating from this hole will be moving at the speed of light *c*. Also, the radiation will be uniformly distributed throughout the hemisphere of solid angles (2π steradians), and one half of the energy will be oriented such that it can move outward through the hole. The spectral radiation intensity is defined as the rate of energy emitted per unit area per unit solid angle and per unit wavelength. The rate of energy emitted per area is simply the product of the energy density derived above and the speed of light (*i.e.*, the distance swept by a ray per unit of time). Therefore, the spectral intensity becomes

$$ I(\lambda, T) = \frac{1}{2} \left[\frac{u(\lambda, T) c}{2 \pi} \right] = \frac{2 h c^2}{\lambda^5} \frac{1}{\exp \left(\frac{hc}{\lambda k_B T} \right) - 1}$$

Similarly, the spectral intensity (per unit frequency instead of wavelength) is

$$ I(\nu, T) = \frac{1}{2} \left[\frac{u(\nu, T) c}{2 \pi} \right]= \frac{2 h \nu^3}{c^2} \frac{1}{\exp \left(\frac{h\nu}{k_B T} \right) - 1}$$